Polynomials and the Intermediate Value Theorem

A polynomial function is one that can be written in the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, \ldots, a_0$ are constants and n is a positive integer.

1. Which of the following are polynomial functions?

   $x \qquad |x| \qquad x^2 \qquad (x-2)^2 \qquad \dfrac{1}{x} \qquad \dfrac{x-3}{2} \qquad \sqrt{x} \qquad 3x^3 + 2^x + 5 \qquad 4 - 2x - x^4 \qquad 3e^x$

2. A polynomial's behavior is determined in large part by its roots. The roots of a polynomial are also known as the zeros of that polynomial, and are the values at which the polynomial equals zero.
   (a) What are the roots of $x^2 - 6x - 7$?

   (b) What are the roots of $(3-x)(2x+1)(x-8)$?

3. If a polynomial has roots $-2, 5$, and 17, what can you say about its graph (if anything)?

4. Write a polynomial that has roots $-2, 5$, and 17.

Even better than knowing the roots of a polynomial is knowing the roots and the *multiplicities* of each root. The multiplicity of a root is how many times it appears as a root of a polynomial. For example, in the polynomial $p(x) = (x-2)(x+3)(x+3)(x-2)(x-5)(x-2)$, the root 2 has multiplicity 3, the root -3 has multiplicity 2, and the root 5 has multiplicity 1. Of course, this polynomial might more commonly be written $p(x) = (x-2)^3(x+3)^2(x-5)$.

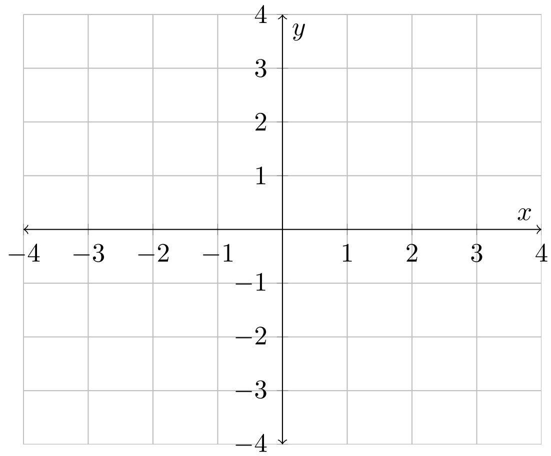
5. Let's determine the effect that the multiplicity of a root has on the graph of a polynomial.
   (a) Let $p_1(x) = x$. What are the roots of p_1, and what are their multiplicities?

   (b) Let $p_2(x) = x^2$. What are the roots of p_2, and what are their multiplicities?

   (c) Let $p_3(x) = x^3$. What are the roots of p_3, and what are their multiplicities?

   (d) Let $p_4(x) = x^4$. What are the roots of p_4, and what are their multiplicities?

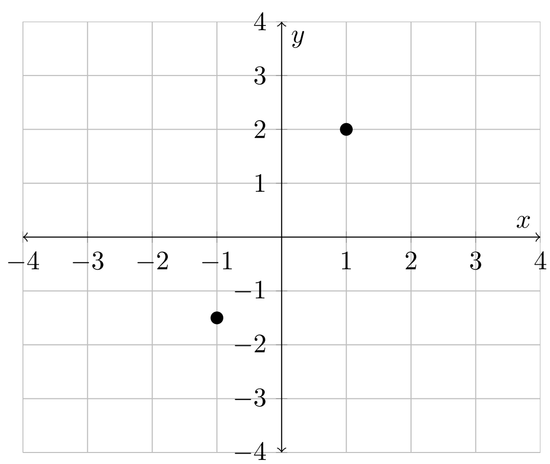
(e) On the axes below, graph each of the above functions. If you're not sure what the graph looks like, plug in values and plot points.



6. How does the multiplicity of a root correspond to what happens in the graph of a polynomial?

Often, we can't find the roots of a polynomial exactly. (The ones you've seen so far have all had nice roots, but what if a polynomial had a root of $x = 3.76168491327$?) However, we can know that roots exist even if we can't find them exactly by using the Intermediate Value Theorem.

7. Below are two points. Connect these two points with a continuous curve, but don't go outside the box and don't cross the x-axis.



If you managed to do that, congratulations – you just bent space-time to your will. If you found it was impossible to do, then also congratulations – you just illustrated the Intermediate Value Theorem. Here's what it says:

Intermediate Value Theorem. Suppose f is a continuous function on an interval containing a and b. Then, for every value u between $f(a)$ and $f(b)$, there exists some c in $[a, b]$ such that $f(c) = u$.

8. Interpret the IVT in your own words. What does it mean?

All polynomial functions are continuous. We'll just take that as a fact.

9. Let's say p is a polynomial function and $p(3) = -2$ and $p(5) = 8$. What can you say about the roots of p?

10. Let $q(x) = x^{11} - 2x^2 + 4$. Find an interval where q is guaranteed to have a root, and explain how you know.

11. Does the IVT work if a function is not continuous? Find a function to show that it does not.

The last thing we'll consider is the *end behavior* of polynomials. That is, does $p(x)$ get really big as x gets big, or really negatively big, or what?

12. Let $p(x) = x^3 - 2x + 5$.
    (a) What is $p(3)$?
    (b) What is $p(10)$?
    (c) What is $p(100)$?
    (d) As x gets bigger, what happens to $p(x)$? That is, as $x \to \infty, p(x) \to ??$.
    (e) What is $p(-10)$? $p(-100)$? As $x \to -\infty, p(x) \to ???$
    (f) Does the $-2x + 5$ part of p really matter for this?

13. Let $q(x) = 3x^4 - x^2 + 5x + 78$.
    (a) As $x \to \infty, p(x) \to$______.

    (b) As $x \to -\infty, p(x) \to$______.

14. Can you find a polynomial p such that the following are satisfied?

    i As $x \to \infty, p(x) \to -\infty$, and
    ii As $x \to -\infty, p(x) \to \infty$.

    If you're stuck, you might try thinking about Question **??** and graph transformations.